Elliptic partial differential equations from an elementary viewpoint

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This is just a short and informal *pre-enrolment quiz*. These are perhaps quite *challenging* calculus questions, so don't get discouraged if you don't solve them right away. Rather, take this as a useful opportunity to revisit and consolidate your calculus skills — and possibly to recast your knowledge into a more advanced and mature display.

No worries: the exercises for the unit will be technically **much simpler** than these (many of them will be provided with hints, and the solutions of the exercises can be easily found in the notes of the course).

(1) Let a_k be a sequence of real numbers and

$$s_k := \sum_{j=0}^{k} a_j.$$

Prove that

$$|a_k| \le 2 \max \{ |s_{k-1}|, |s_k| \}.$$

(2) Compute

$$\lim_{x \to +\infty} \left\{ \frac{x}{2e} + x^2 \left[\frac{1}{e} - \left(\frac{x}{x+1} \right)^x \right] \right\}.$$

(3) Prove that the sequence of functions

$$f_n(x) := \frac{1}{\exp(x) + \exp(x^2) + \cdots + \exp(x^n)}$$

converges to zero uniformly in $\mathbb{R}$.

SOLUTIONS

(1) Notice that

$$s_k = \sum_{j=0}^{k-1} a_j + a_k = s_{k-1} + a_k$$

and therefore $a_k = s_k - s_{k-1}$. It follows that

$$\begin{aligned}|a_k| = |s_k - s_{k-1}| &\le |s_k| + |s_{k-1}| \\ &\le \max\big\{|s_{k-1}|,\, |s_k|\big\} + \max\big\{|s_{k-1}|,\, |s_k|\big\} \\ &= 2\max\big\{|s_{k-1}|,\, |s_k|\big\}.\end{aligned}$$

This was an exercise on sequences and summations. The solution was short but tricky. It should help you develop some feeling with algebraic computations.

(2) The answer is

$$\frac{5}{24\, e}.$$

To see this, we let $t := \frac{1}{x}$ and we observe that $t \to 0^+$. Therefore,

$$\begin{aligned}\left(\frac{x}{x+1}\right)^x &= \exp\left(x \ln\left(\frac{x}{x+1}\right)\right) = \exp\left(\frac{1}{t}\ln\left(\frac{\frac{1}{t}}{\frac{1}{t}+1}\right)\right) \\ &= \exp\left(\frac{1}{t}\ln\left(\frac{1}{1+t}\right)\right).\end{aligned}$$

Now, since

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + O(t^4)$$

we have that

$$\ln\left(\frac{1}{1+t}\right) = \ln\Big(1 - t + t^2 - t^3 + O(t^4)\Big).$$

Also, calling $\tau := t - t^2 + t^3 + O(t^4)$,

$$\begin{aligned}&\ln(1-\tau) \\ =\ & -\tau - \frac{\tau^2}{2} - \frac{\tau^3}{3} + O(\tau^4) \\ =\ & -t + t^2 - t^3 - \frac{\big(t - t^2 + t^3 + O(t^4)\big)^2}{2} - \frac{\big(t - t^2 + t^3 + O(t^4)\big)^3}{3} + O(t^4)\end{aligned}$$

$$= \; -t + t^2 - t^3 - \frac{t^2 - 2t^3}{2} - \frac{t^3}{3} + O(t^4)$$
$$= \; -t + \frac{t^2}{2} - \frac{t^3}{3} + O(t^4).$$

Collecting these items of information, we find

$$\begin{aligned}\left(\frac{x}{x+1}\right)^x &= \exp\left(\frac{1}{t}\left(-t + \frac{t^2}{2} - \frac{t^3}{3} + O(t^4)\right)\right) \\ &= \exp\left(-1 + \frac{t}{2} - \frac{t^2}{3} + O(t^3)\right) \\ &= \frac{\exp\left(\frac{t}{2} - \frac{t^2}{3} + O(t^3)\right)}{e}.\end{aligned}$$

Thus, calling $\vartheta := \frac{t}{2} - \frac{t^2}{3} + O(t^3)$ and using that

$$e^\vartheta = 1 + \vartheta + \frac{\vartheta^2}{2} + O(\vartheta^3),$$

we see that

$$\exp\left(\frac{t}{2} - \frac{t^2}{3} + O(t^3)\right) = 1 + \frac{t}{2} - \frac{t^2}{3} + \frac{\left(\frac{t}{2} - \frac{t^2}{3}\right)^2}{2} + O(t^3)$$
$$= 1 + \frac{t}{2} - \frac{t^2}{3} + \frac{t^2}{8} + O(t^3) = 1 + \frac{t}{2} - \frac{5\,t^2}{24} + O(t^3).$$

Consequently,

$$\left(\frac{x}{x+1}\right)^x = \frac{1}{e} + \frac{t}{2e} - \frac{5\,t^2}{24\,e} + O(t^3)$$

and, as a byproduct,

$$\frac{x}{2e} + x^2\left[\frac{1}{e} - \left(\frac{x}{x+1}\right)^x\right] = \frac{1}{2e\,t} + \frac{1}{t^2}\left[-\frac{t}{2e} + \frac{5\,t^2}{24\,e} + O(t^3)\right]$$
$$= \frac{5}{24\,e} + O(t),$$

which gives that the desired limit is equal to $\frac{5}{24\,e}$.

This was a complicated exercise on Taylor expansions. It should help you revisit Taylor expansions and Taylor series and develop a feeling on the notion of infinitesimals (and also, to exercise your patience and concentration on a rather long computation).

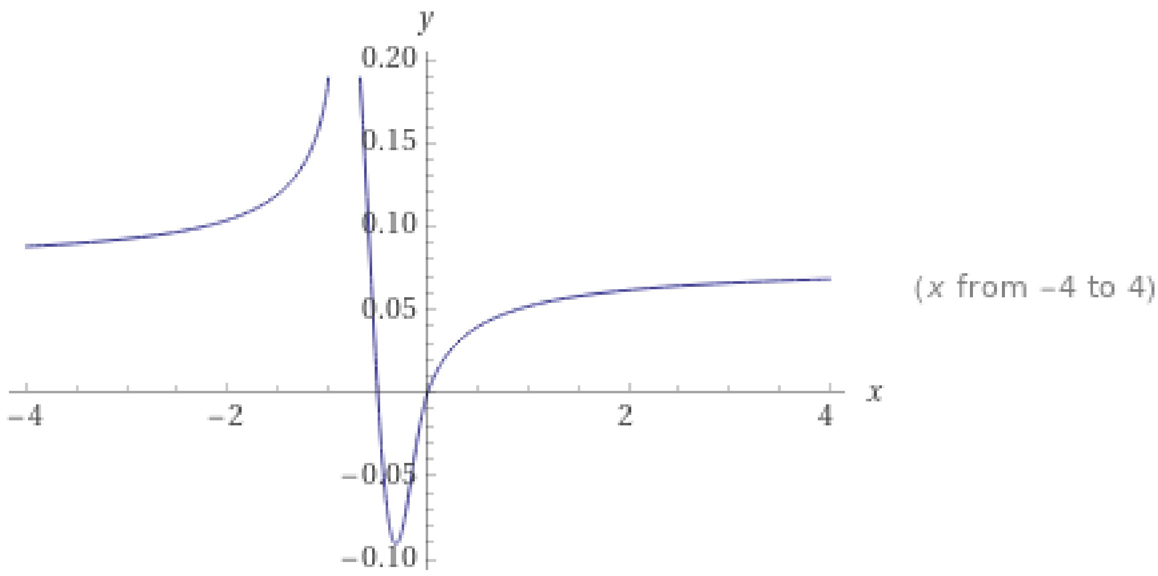

FIGURE 1. Plot of the function $y = \frac{x}{2e} + x^2 \left[\frac{1}{e} - \left(\frac{x}{x+1}\right)^x\right]$.

Of course, it is a bit tedious to do computations and, in a sense, computers can (sometimes) do exercises like this faster and better: for instance Wolfram will tell you right away the expansion at $x = +\infty$

$$\begin{aligned} &\frac{x}{2e} + x^2 \left[\frac{1}{e} - \left(\frac{x}{x+1}\right)^x\right] \\ = \ &\frac{5}{24\,e} - \frac{5}{48\,e\,x} + \frac{337}{5760\,e\,x^2} - \frac{137}{3840\,e\,x^3} + O\left(\frac{1}{x^4}\right), \end{aligned}$$

from which the desired limit follows immediately, and plot for you the function in Figure 1: but as soon as things get a bit more interesting, for instance involving functions rather than numbers, or objects we know only up to a certain extent, Wolfram can easily become silent or produce unreliable results, therefore it is a good training to consolidate our technical skills and master the notions of infinite and infinitesimal (though it takes some consistency and ability to endure difficulties to do so).

(3) We write $n = 2m + j$ with $j \in \{0, 1\}$ (that is, if n is odd we take $m := \frac{n-1}{2}$ and $j = 1$, if n is even we take $m := \frac{n}{2}$ and $j = 0$).

In any case,

$$\exp(x) + \exp(x^2) + \cdots + \exp(x^n) \geq \sum_{i=1}^{2m} \exp(x^i)$$

$$\geq \sum_{k=1}^{m} \exp(x^{2k}) \geq \sum_{k=1}^{m} 1 = m \geq \frac{n-1}{2}.$$

As a result,

$$0 \leq f_n(x) \leq \frac{2}{n-1},$$

from which we conclude that f_n converges to zero uniformly in $\mathbb{R}$.

This exercise helped you review the notion of uniform convergence for sequences of functions. Also, it developed some algebraic skills related to odd and even numbers and recalled some basic properties of the exponential function.

Suggestion

If you solved these exercises right away, good for you, **be very proud of yourself**. If not, **don't panic**, these are perhaps not the standard calculus exercises you may have been repeatedly exposed to! Just take this as a learning opportunity and try to proceed like this:

- pick one exercise you couldn't do,
- have a look to the solution and see if, with this additional hint, you can make any further progress yourself,
- if not, check the solution carefully and try to understand it in detail (to understand it, not to memorize it!),
- now try to redo the exercise yourself (either using the solution you have read and understood, or finding your own way).

Yes, it may take a bit of time, but this is the fun of the learning process ☺☻