Problems

1. Consider the data points $\{(-1,2),(0,6),(1,21),(2,70)\}$.

   (a) Write the interpolation polynomial using a monomial basis (Vandermonde form). *The monomial basis for the polynomial space of degree n is $\{1, x, x^2, \cdots, x^n\}$.* We write

   $$\mathcal{P}_n(\mathbb{R}) = \text{span}\{1, x, x^2, \cdots, x^n\}.$$

   (b) Write the interpolation polynomial using a Lagrange basis (Lagrange form). Verify that it is the same polynomial as in (a) by converting to monomial basis form.

   Definition 1. *A set of polynomial functions $\{l_i\}_{i=0}^n$ is said to be a* Lagrange basis *or* cardinal basis *for the space of polynomials of degree n with respect to the set of distinct points $\{x_i\}_{i=0}^n$ if $l_i(x_j) = \delta_{ij}$. We have*

   $$\mathcal{P}_n(\mathbb{R}) = \textit{span}\{l_1(x), l_2(x), l_3(x), \cdots, l_n(x)\}.$$

2. What is the largest k for the function $f(x) = |x|$ so that $f \in C^k(\mathbb{R})$?

3. Let $\Omega = (0,1) \times (0,1)$ and $u : \Omega \to \mathbb{R}$ be defined as

   $$u(x,y) = x^2 + y^2.$$

   Compute the norms $\|u\|_{C^1(\bar{\Omega})}$ and $\|u\|_{L^2(\Omega)}$.

   Hint: If $\Omega \subset \mathbb{R}^d$ is a bounded open set, $C^k(\bar{\Omega})$ denotes the set of all $u \in C^k(\Omega)$ such that $D^\alpha u$ can be extended to a continuous function on $\bar{\Omega}$, the closure of Ω, for all

   $$\alpha = (\alpha_1, \cdots, \alpha_d) \quad \textit{with} \quad |\alpha| \le k.$$

   We can use the following norm for functions in $C^k(\bar{\Omega})$:

   $$\|u\|_{C^k(\bar{\Omega})} = \sum_{|\alpha| \le k} \sup |D^\alpha u(x_1, \cdots, x_d)|.$$

   It is standard to write $C(\bar{\Omega})$ for $C^0(\bar{\Omega})$ when $k = 0$. Also note that

   $$\|u\|_{L^2(\Omega)} = \sqrt{\int_\Omega u^2(x_1, \cdots, x_d) dx_1 \, dx_2 \cdots dx_d}.$$

4. Derive Simpson's rule with $O(h^5)$ error term by using the fact that the rule is exact for x^n when $n = 1, 2,$ and 3 to determine a_0, a_1, and a_2 in the formula

   $$\int_{x_0}^{x_2} f(x)\, dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi),$$

   where $x_1 = (x_0 + x_2)/2$ and ξ is any suitable number. Then find k by applying this integration formula for $f(x) = x^4$. The following numerical integration rule is called the Simpson's rule:

   $$\int_a^b g(y)\, dy \approx \frac{(b-a)}{6} \left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b) \right].$$

5. Let $f \in C^4(a,b)$ and $x_0 \in (a,b)$. Derive a formula to approximate $f'(x)$ and $f''(x)$ at $x = x_0$ using $f(x_0 - h)$, $f(x_0)$, $f(x_0 + h)$ where $x_0 - h, x_0 + h \in (a,b)$. *Hint: consider the expression*

$$Af(x_0 - h) + Bf(x_0 + h) + Cf(x_0),$$

and expand all terms in Taylor polynomials of suitable order about x_0, and compare with $f'(x_0)$ and $f''(x_0)$ What are the truncation errors in these formulas?

6. The sine function has the power series definition

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Write a function `SineTaylor.m` that has input n and x and output the relative error in the partial sums

$$S_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

The relative error R_n is then defined as

$$R_n = \frac{|\sin(x) - S_n(x)|}{|\sin(x)|}.$$

Think of an efficient way of computing S_n. You can start your function file in the following way:

```
function relErr = SineTaylor(x,n)
% SineTaylor(x,n) evaluates the relative error between sin(x)
% and its n-term MacLaurin series at x
% Usage: relErr=SineTaylor(x,n)
%inputs: x point where we want to compute the error
%        n number of terms in MacLaurin series
%output: relErr the relative error in approximation
```

Plot the errors for $n = 2, 4, 6, 8, 10, 12, 14, 16, 18, 20$ using $x = 4$ using the logarithmic scale. What happens when n is large?

Solutions

1. (a) We want to find a cubic polynomial $p_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ that interpolates the data. Each interpolation point leads to a linear equation relating the four unknowns a_0, a_1, a_2, and a_3:

$$\begin{aligned} p_3(-1) = 2 &\Rightarrow a_0 - a_1 + a_2 - a_3 = 2 \\ p_3(0) = 6 &\Rightarrow a_0 = 6 \\ p_3(1) = 21 &\Rightarrow a_0 + a_1 + a_2 + a_3 = 21 \\ p_3(2) = 70 &\Rightarrow a_0 + 2a_1 + 4a_2 + 8a_3 = 70. \end{aligned}$$

These equations can be expressed in a matrix/vector form

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 21 \\ 70 \end{bmatrix}$$

The polynomial in the monomial form is

$$p(x) = 6 + 17/3x + 11/2x^2 + 23/6x^3.$$

(b) The four Lagrange basis are given by

$$l_1(x) = -\frac{x\ (x-1)\ (x-2)}{6},\ l_2(x) = \frac{(x-1)\ (x+1)\ (x-2)}{2},$$
$$l_3(x) = -x\left(\frac{x}{2} + \frac{1}{2}\right)(x-2),\ l_4(x) = \frac{x\ (x^2-1)}{6}.$$

The required Lagrange interpolation polynomial is

$$p(x) = 2l_1(x) + 6l_2(x) + 21l_3(x) + 70L_4(x).$$

A simplification yields

$$p(x) = 6 + 17/3x + 11/2x^2 + 23/6x^3.$$

2. Here, the function is only continuous not even differentiable at $x = 0$. Thus, $k = 0$.

3. 6 and $\sqrt{\frac{28}{45}}$

4. Let $h = x_1 - x_0$. Thus $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$. Using the fact that the formula is exact x^n with $n = 1, 2, 3$, we have three equations

$$\begin{aligned} a_0x_0 + a_1(x_0+h) + a_2(x_0+2h) &= 2x_0h + 2h^2 \quad [\text{using } f(x) = x] \\ a_0x_0^2 + a_1(x_0+h)^2 + a_2(x_0+2h)^2 &= 2x_0^2h + 4x_0h^2 + \frac{8h^3}{3} \quad [\text{using } f(x) = x^2] \\ a_0x_0^3 + a_1(x_0+h)^3 + a_2(x_0+2h)^3 &= 2x_0^3h + 6x_0^2h^2 + 8x_0h^3 + 4h^4 \quad [\text{using } f(x) = x^3]. \end{aligned}$$

Solving these equations for a_0, a_1 and a_2, we get

$$a_0 = h/3,\ a_1 = 4h/3,\ a_2 = h/3.$$

Using $f(x) = x^4$, we get

$$\frac{1}{5}(x_2^5 - x_0^5) = \frac{h}{3}(x_0^4 + 4x_1^4 + x_2^4) + 24k.$$

Hence

$$k = \frac{1}{120}(x_2^5 - x_0^5) - \frac{h}{72}(x_0^4 + 4x_1^4 + x_2^4).$$

Using $x_1 = (x_0 + h)$ and $x_2 = (x_0 + 2h)$ we get

$$k = \frac{1}{120}(x_0^5 + 10hx_0^4 + 40h^2x_0^3 + 80h^3x_0^2 + 80h^4x_0 + 32h^5 - x_0^5) - \frac{h}{72}\left[x_0^4 + 4(x_0^4 + 4hx_0^3 + 6h^2x_0^2 + 4h^3x_0 + h^4) + (x_0^4 + 8hx_0^3 + 24h^2x_0^2 + 32h^3x_0 + 16h^4)\right].$$

This simplifies to $k = -h^5/90$.

5. From Taylor's Remainder theorem, we have

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(\xi)$$
$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(\xi)$$

We combine these equations to get

$$\frac{1}{h^2}[f(x_0 + h) + f(x_0 - h) - 2f(x_0)] = f''(x_0) + O(h^2)$$

and

$$\frac{1}{2h}[f(x_0 + h) - f(x_0 - h)] = f'(x_0) + O(h^2).$$

Here we have used the big "O" notation.

6. Include the following lines of code in the function defintion.

```
r1=[0:1:n];
r2=(-1).^(r1);
Sn=sum(r2.*(x*(x.^(2*r1))./factorial(2*r1+1)));
relErr=abs(sin(x)-Sn)/abs(sin(x));
```

You can use the following piece of MATLAB code to plot the result.

```
idx =2:2:20;
for j=1:length(idx)
RE(j) =SineTaylor(4,idx(j));
end
loglog(idx,RE,'-*','markersize',20);
```