ACE Network Subject Information Guide

Algebraic Number Theory

Semester 1, 2025

Administration and contact details

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Host institution	University of Newcastle
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Subject details

Handbook entry URL	N/A
Subject homepage URL	N/A
Honours student hand-out URL	N/A (PDF available on request)
Teaching period (start and end date):	24 Feb - 30 May 2025
Exam period (start and end date):	10 – 20 June 2025
Contact hours per week:	2
ACE enrolment closure date:	10 March 2025
Lecture day(s) and time(s):	Mondays 13h00-15h00
Description of electronic access arrangements for students (for example, LMS)	Canvas access at the following link: https://catalog.newcastle.edu.au/browse/cese/courses/math4102-directed-studies-ii-s1-2025-callaghan

Subject content

1. Subject content description

Number theory is the study of the integers. However, in order to study questions about the integers, one is often forced to study more general sets of numbers. For example, in order to determine which prime numbers can be written as the sum of two squares, $p = x^2 + y^2$, one really needs to consider numbers of the form $x + iy$, which are irrational but algebraic.

Algebraic Number Theory is thus the study of algebraic numbers (i.e. solutions to polynomial equations with integer coefficients). These are elements in number fields, i.e. finite extensions of the field of rational numbers, for example $\mathbb{Q}(i)$, the Gaussian numbers.

Each such number field contains a subring of algebraic integers (e.g. $\mathbb{Z}[i]$, the Gaussian integers in our example above) and we're interested in arithmetic in these rings of algebraic integers. In the Gaussian integers, every element can be factorised uniquely into a product of prime elements. However, in many other examples, this unique factorisation fails, for example in the ring $\mathbb{Z}[\sqrt{-5}]$.

This obstacle can be overcome by moving from elements to ideals – it turns out that every ideal in an algebraic number ring can be factorised uniquely into a product of prime ideals.

This is the starting point for a very rich theory in which we will study these rings (more precisely, a class of rings called Dedekind rings), their groups of units and ideal classes modulo principal ideals, which form a finite group called the ideal class group.

Number theory is famous for borrowing techniques from all other branches of mathematics. The exact topics we will study will determine the techniques we will use, and will be decided based on the interests and backgrounds of the students.

2. Week-by-week topic overview

Topics covered will include the following:
- Number fields
- Dedekind rings, unique factorisation of ideals into prime ideals
- Minkowski's geometry of numbers and finiteness of the ideal class group
- Dirichlet's Unit Theorem.

Many examples will be considered throughout, especially from quadratic fields and cyclotomic fields. We will also use the free Mathematics software package SageMath (https://www.sagemath.org/) to compute various examples.

Optional topics may include, depending on time and student interest:

- Analytic number theory: zeta functions and the distribution of primes
- L-functions, primes in arithmetic progressions and Dirichlet's class number formulas
- Valuation theory and p-adic numbers
- Galois theory in number fields
- Algebraic function fields
- Elliptic curves

In this course, students are expected to read the relevant sections of the notes themselves in preparation for each session. Contact time will be used to discuss and clarify this content.

3. Assumed prerequisite knowledge and capabilities

A basic understanding of groups, rings, fields and ideals, such as is taught in a standard undergraduate Abstract Algebra course is assumed.
A first course in Number Theory would be helpful, but is not necessary.

4. Learning outcomes and objectives

- Demonstrate an understanding of the content and context of algebraic number theory
- Apply advanced mathematical problem solving skills
- Present coherent mathematical arguments in written form.

AQF specific Program Learning Outcomes and Learning Outcome Descriptors (if available):

AQF Program Learning Outcomes addressed in this subject	**Associated AQF Learning Outcome Descriptors for this subject**
Insert Program Learning Outcome here	Choose from list below
Insert Program Learning Outcome here	Choose from list below
Insert Program Learning Outcome here	Choose from list below
Insert Program Learning Outcome here	Choose from list below
Insert Program Learning Outcome here	Choose from list below
Insert Program Learning Outcome here	Choose from list below
Insert Program Learning Outcome here	Choose from list below

Learning Outcome Descriptors at AQF Level 8 **Knowledge** K1: coherent and advanced knowledge of the underlying principles and concepts in one or more disciplines K2: knowledge of research principles and methods **Skills** S1: cognitive skills to review, analyse, consolidate and synthesise knowledge to identify and provide solutions to complex problem with intellectual independence S2: cognitive and technical skills to demonstrate a broad understanding of a body of knowledge and theoretical concepts with advanced understanding in some areas S3: cognitive skills to exercise critical thinking and judgement in developing new understanding S4: technical skills to design and use in a research project S5: communication skills to present clear and coherent exposition of knowledge and ideas to a variety of audiences **Application of Knowledge and Skills** A1: with initiative and judgement in professional practice and/or scholarship A2: to adapt knowledge and skills in diverse contexts A3: with responsibility and accountability for own learning and practice and in collaboration with others within broad parameters A4: to plan and execute project work and/or a piece of research and scholarship with some independence

5. Learning resources

The course will follow lecture notes by Matthew Baker (https://sites.google.com/view/mattbakermath/publications#h.uuedvamjgl11)

There are many good books on Algebraic Number Theory, I recommend in particular:

- Paul Pollack: "A Conversational Introduction to Algebraic Number Theory", AMS Student Mathematical Library, vol 84.
- Pierre Samuel: "Algebraic Theory of Numbers", Dover

More advanced textbooks include:

- Serge Lang: "Algebraic Number Theory", Springer Graduate Texts in Mathematics vol 110.
- Jürgen Neukrich: "Algebraic Number Theory", Springer
- James Milne: "Algebraic Number Theory", free course notes at https://www.jmilne.org/math/CourseNotes/ant.html

6. Assessment

Exam/assignment/classwork breakdown					
Exam	50 %	**Assignment**	50 %	**Class work**	0 %
Assignment due dates		4 April 2025	30 May 2025		
Approximate exam date				10 June 2025	

Institution honours program details

Weight of subject in total honours assessment at host department	10 units of 80 total
Thesis/subject split at host department	40 units of 80 total
Honours grade ranges at host department	
H1	85 - 100 %
H2a	75 - 84 %
H2b	65 - 74 %
H3	50 - 64 %

Institution masters program details

Weight of subject in total masters assessment at host department	Click here to enter text.
Thesis/subject split at host department	Click here to enter text.
Masters grade ranges at host department	
H1	Enter range %
H2a	Enter range %
H2b	Enter range %
H3	Enter range %