Diagnostic test + Solution

Problem Statement

Suppose that X_i are independent and identically distributed positive random variables with mean 1. The aim of this problem is to find the (a.s.) limit of

$$M_n = \prod_{i=1}^{n} X_i$$

as n grows. In the following we remove a.s. from the statement of the problem.

If X_i are deterministically equal to one, there is nothing to prove (why?). Suppose that $\mathrm{Var}(X_i) > 0$.

- Prove that $\mathbb{E}[\log X_i] < 0$ using Jensen's inequality.
- Prove that
$$\lim_{n\to\infty} \sum_{i=1}^{n} \log X_i = -\infty.$$
- Use the previous two items to prove that M_n converges to zero if $\mathrm{Var}(X_1) > 0$.
- Interpret your result.

Solution

Step 1: Proving that $\mathbb{E}[\log X_i] < 0$ using Jensen's inequality

Since X_i are positive random variables with mean $\mathbb{E}[X_i] = 1$, we consider the function $f(x) = \log x$, which is a strictly concave function for $x > 0$. By Jensen's inequality, we have:

$$\mathbb{E}[\log X_i] < \log(\mathbb{E}[X_i]) = \log(1) = 0.$$

Since $\mathrm{Var}(X_i) > 0$, and since $\log x$ is strictly concave, we have a strict inequality above. In summary:

$$\mathbb{E}[\log X_i] < 0.$$

Step 2: Proving that $\sum_{i=1}^{n} \log X_i \to -\infty$

Define S_n as the sum of logarithms:

$$S_n = \sum_{i=1}^{n} \log X_i.$$

Since X_i are independent and identically distributed (iid), by the Law of Large Numbers (LLN), we have:

$$\frac{S_n}{n} \to \mathbb{E}[\log X_i] < 0 \quad \text{almost surely as } n \to \infty.$$

which implies

$$S_n = \sum_{i=1}^{n} \log X_i \to -\infty \quad \text{almost surely.}$$

Step 3: Proving that $M_n \to 0$

Since M_n is the product of the X_i, we take the logarithm:

$$\log M_n = \sum_{i=1}^{n} \log X_i = S_n.$$

We know that $S_n \to -\infty$. Taking the exponential on both sides:

$$M_n = e^{S_n} \to e^{-\infty} = 0.$$

Thus, M_n converges to 0 almost surely.

Step 4: Interpretation

Intuition behind this result is that $(1-\epsilon)(1+\epsilon) < 1$ while $(1-\epsilon+1+\epsilon)/2 = 1$.