Representation Theory — Pre-Quiz

(1) Let G be an abelian group and let $H = \{x^2 \mid x \in G\}$. Show that H is a subgroup of G.

(2) Let $\mathbb{R}$ denote the group of real numbers under addition and S^1 denote the group $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication. Let $\Theta : \mathbb{R} \to S^1$ be the map defined by $(x)\Theta = e^{xi}$.

  (a) Show that Θ is a homomorphism.

  (b) What is the kernel of Θ?

  (c) State the First Isomorphism Theorem.

(3) Let G be a group and $g \in G$. Show that the map $\iota_g : G \to G$ such that $(x)\iota_g = g^{-1}xg$ is an automorphism of G.

(4) Let G be an abelian group with normal subgroup N. Show that the quotient group G/N is abelian.

(5) Let A be an $n \times n$ matrix with entries from the field F. Show that any conjugate $T^{-1}AT$ of A for $T \in \mathrm{GL}(n, F)$, has the same set of eigenvalues as A.

Solutions

(1) Note that $e = e^2 \in H$ so H is nonempty. Let $a, b \in H$. Then $a = x^2$ for some $x \in G$ and $b = y^2$ for some $y \in G$. Also

$$ab^{-1} = x^2(y^2)^{-1} = x^2(y^{-1})^2 = xy^{-1}xy^{-1} = (xy^{-1})^2 \in H$$

where the second last equality holds as G is abelian. Thus by the 2nd Subgroup Test, H is a subgroup of G.

(2) (a) Let $x, y \in \mathbb{R}$. Then $(x+y)\Theta = e^{(x+y)i} = e^{xi}e^{yi} = (x)\Theta(y)\Theta$. Thus Θ is a homomorphism.

(b) $\ker(\Theta) = \{x \in \mathbb{R} \mid e^{xi} = 1\} = \{2\pi n \mid n \in \mathbb{Z}\}$.

(c) Let G and H be groups, and $\varphi : G \to H$ be a homomorphism. Then $G/\ker(\varphi) \cong (G)\varphi = \{(g)\varphi \mid g \in G\}$

(3) Let $x, y \in G$. Then $(xy)\iota_g = g^{-1}(xy)g = g^{-1}xeyg = g^{-1}xgg^{-1}yg = (x)\iota_g(y)\iota_g$. Hence ι_g is a homomorphism from G to G. Suppose that $x \in \ker(\iota_g)$. Then $g^{-1}xg = e$ and so $x = e$. Thus $\ker(\iota_g) = \{e\}$ and so ι_g is one-to-one. Finally, suppose that $x \in G$. Then $(gxg^{-1})\iota_g = x$ and so ι_g is onto. Thus ι_g is an automorphism of G.

(4) Let $Na, Nb \in G/N$. Then $(Na)(Nb) = N(ab) = N(ba) = (Nb)(Na)$, where the second last equality is since G is abelian. Hence G/N is abelian.

(5) Let v be an eigenvector for A with eigenvalue λ and consider the vector vT. Then

$$\begin{aligned}(vT)(T^{-1}AT) &= v(TT^{-1})AT \\ &= (vA)T \\ &= (\lambda v)T \\ &= \lambda(vT).\end{aligned}$$

Hence vT is an eigenvector for $T^{-1}AT$ with eigenvalue λ. Conversely, if w is a eigenvector for $T^{-1}AT$ with eigenvalue λ then vT^{-1} is an eigenvector for A with eigenvalue λ. Thus A and $T^{-1}AT$ have the same set of eigenvalues.