ACE Elliptic Functions, Elliptic Curves and Modular Forms – Pre-Quiz

Questions

1. Determine all $\omega \in \mathbb{C}$ for which $e^{\omega+z} = e^z$ holds for all $z \in \mathbb{C}$.

2. Compute the contour integral
$$\oint_C \frac{dz}{\sin z},$$
where C is the unit circle $|z| = 1$.

3. Let f be a meromorphic function on an open set containing a simple closed curve Γ and its interior, and suppose f has no zeroes or poles on Γ. Write down an expression (involving a contour integral) which counts (with multiplicity) the number of zeroes and poles of f inside Γ.

Solutions

1. We are given that
$$e^{\omega+z} = e^z \qquad \text{for all } z \in \mathbb{C}.$$
Using the exponential law,
$$e^{\omega+z} = e^\omega e^z,$$
so the identity becomes
$$e^\omega e^z = e^z \qquad \text{for all } z.$$
Rearranging gives
$$(e^\omega - 1)e^z = 0 \qquad \text{for all } z.$$
Since $e^z \neq 0$ for any $z \in \mathbb{C}$, we must have
$$e^\omega = 1.$$
The complex solutions of this equation are precisely
$$\omega = 2\pi i k, \qquad k \in \mathbb{Z}.$$
$$\boxed{\omega \in 2\pi i\mathbb{Z}}.$$

2. Let C be the unit circle $|z| = 1$. The integrand
$$f(z) = \frac{1}{\sin z}$$
is meromorphic with simple poles at the zeros of $\sin z$, i.e. at $z = n\pi$ for $n \in \mathbb{Z}$. Inside the unit circle, the only such pole is $z = 0$.

Near $z = 0$ we have the Taylor expansion
$$\sin z = z + O(z^3),$$
so
$$\frac{1}{\sin z} = \frac{1}{z} + O(z).$$
Thus the residue at $z = 0$ is
$$\operatorname{Res}\left(\frac{1}{\sin z}, 0\right) = 1.$$
By the residue theorem,
$$\oint_C \frac{dz}{\sin z} = 2\pi i \cdot \operatorname{Res}\left(\frac{1}{\sin z}, 0\right) = 2\pi i.$$
$$\boxed{\oint_C \frac{dz}{\sin z} = 2\pi i.}$$

3. Let f be meromorphic in a region containing a simple closed curve Γ and its interior, and suppose that f has no zeros or poles on Γ.

The argument principle states that
$$\frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)}\, dz = N - P,$$
where

- N is the number of zeros of f inside Γ, counted with multiplicity;
- P is the number of poles of f inside Γ, counted with multiplicity.

In particular, if f is holomorphic inside Γ (so $P = 0$), this integral directly counts the zeros:

$$\boxed{N = \frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)}\, dz.}$$

More generally,

$$N = P + \frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)}\, dz.$$